

Zeta and normal zeta functions for a subclass of space groups

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Abstract We calculate zeta and normal zeta functions of space groups with the point group isomorphic to the cyclic group of order 2. The obtained results are applied to determine the number of subgroups, resp. normal subgroups, of a given index for each of these groups.

Keywords Zeta functions of groups · Subgroup growth · Crystallographic groups

Mathematics Subject Classification 11M41 · 20H15

1 Introduction

The concepts of the zeta and normal zeta function of a group were introduced by Grunewald et al. [4] and Smith [7]. The zeta function of a group G is defined as $\zeta_G(s) = \sum_{n \in \mathbb{N}} a_n(G)n^{-s}$, where $a_n(G)$ denotes the number of subgroups of index n in G . The normal zeta function of a group G is given by $\zeta_G^{\triangleleft}(s) = \sum_{n \in \mathbb{N}} c_n(G)n^{-s}$, where $c_n(G)$ is the number of normal subgroups of index n in G . These functions provide a useful tool for studying the relationship between the asymptotic behavior of the sequences $a_n(G)$, resp. $c_n(G)$, and the structure of G .

If a group G has a polynomial subgroup growth, i.e. if $a_n(G) \leq n^k$ for some k and for all $n \in \mathbb{N}$, then $\zeta_G(s)$ has a non-trivial domain of convergence. In particular, if

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G is residually finite nilpotent group then $a_n(G)$ grows polynomially and $\zeta_G(s) = \sum_{n \in \mathbb{N}} a_n(G) n^{-s}$ satisfies an Euler product formula $\zeta_G(s) = \prod_{p \in P} \zeta_{G,p}(s)$, where $\zeta_{G,p}(s)$ counts only subgroups of p -power index and P denotes the set of all primes.

du Sautoy et al. [2] proved the following theorem.

Theorem 1.1 *Let G be a finite extension of a free abelian group of finite rank. Then $\zeta_G(s)$ and $\zeta_G^{\Delta}(s)$ can be extended to meromorphic functions on the whole complex plane.*

Lubotzky and du Sautoy [1] established a functional equation $\zeta_{G,p}(s) \Big|_{p \rightarrow p^{-1}} = (-1)^n p^{as+bn} \zeta_{G,p}(s)$ satisfied by the local factors of the zeta function of a group for some torsion-free nilpotent groups and appropriate $a, b, n \in \mathbb{N}$. Here $p \rightarrow p^{-1}$ denotes a formal inversion of the local parameter p . In this regard, see also [8,9].

There are relatively few explicit expressions known for zeta functions of groups. McDermott [6] calculated the zeta functions of the seventeen plane crystallographic groups.

A space group represents a description of the symmetry of a crystal. A crystallographic group G contains a translation subgroup T which consists of all elements of the group corresponding to translations of the pattern involved. The factor group G/T is known as the point group of G and is denoted by P . Group G is a finite extension of T by P , since P is finite.

In this paper, we calculate zeta and normal zeta functions of space groups with the point group isomorphic to the cyclic group of order 2. There are eight such groups: $P\bar{1}$, $P2$, $P2_1$, $C2$, Pm , Pc , Cm and Cc [5]. After stating results in Sect. 2, we provide the application of these results to compute the number of subgroups of a given index for each group in Sects. 3 and 4. We describe the method of proof in Sect. 5 and provide full details in case of the group $C2$, as a sample.

2 Results

For a sake of brevity, the following notation for translates of the Riemann zeta function is used in the sequel:

$$\zeta_k(s) = \zeta(s - k), \text{ i.e., } \zeta_2(s) = \zeta(s - 2).$$

Theorem 2.1 *Zeta functions of space groups with the point group isomorphic to the cyclic group of order 2 read as follows*

$$\zeta_{P\bar{1}}(s) = \zeta_1(s)\zeta_2(s)\zeta_3(s) + 2^{-s}\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{P2}(s) = (1 + 2^{-s+3})\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{P2_1}(s) = \zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{C2}(s) = (1 + 2^{-2s+3})\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{Pm}(s) = (1 + 9 \cdot 2^{-s} + 6 \cdot 2^{-2s})\zeta(s)\zeta_1(s)\zeta_1(s) + 2^{-s}\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{Pc}(s) = (1 + 2^{-s} - 2 \cdot 2^{-2s})\zeta(s)\zeta_1(s)\zeta_1(s) + 2^{-s}\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{Cm} = (1 + 2^{-s} + 6 \cdot 2^{-2s} + 8 \cdot 2^{-3s})\zeta(s)\zeta_1(s)\zeta_1(s) + 2^{-s}\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{Cc}(s) = (1 - 3 \cdot 2^{-s} + 10 \cdot 2^{-2s} - 8 \cdot 2^{-3s})\zeta(s)\zeta_1(s)\zeta_1(s) + 2^{-s}\zeta(s)\zeta_1(s)\zeta_2(s)$$

Theorem 2.2 Normal zeta functions of space groups with the point group isomorphic to the cyclic group of order 2 are given by

$$\zeta_{P\bar{1}}^{\triangleleft}(s) = 1 + 14 \cdot 2^{-s} + 28 \cdot 2^{-2s} + 8 \cdot 2^{-3s} + 2^{-s}\zeta(s)\zeta_1(s)\zeta_2(s)$$

$$\zeta_{P2}^{\triangleleft}(s) = (1 + 13 \cdot 2^{-s} + 22 \cdot 2^{-2s} + 4 \cdot 2^{-3s})\zeta(s) + (3 \cdot 2^{-2s} + 2^{-s})\zeta(s)\zeta(s)\zeta_1(s)$$

$$\zeta_{P2_1}^{\triangleleft}(s) = (1 + 5 \cdot 2^{-s} - 2 \cdot 2^{-2s} - 4 \cdot 2^{-3s})\zeta(s) + (2^{-s} + 3 \cdot 2^{-2s})\zeta(s)\zeta(s)\zeta_1(s)$$

$$\zeta_{C2}^{\triangleleft}(s) = (2 \cdot 2^{-2s} + 5 \cdot 2^{-s} + 1)\zeta(s) + 2^{-s} \cdot (1 - 2^{-s} + 4 \cdot 2^{-2s})\zeta(s)\zeta(s)\zeta_1(s)$$

$$\zeta_{Pm}^{\triangleleft}(s) = (1 + 11 \cdot 2^{-s} + 12 \cdot 2^{-2s})\zeta(s)\zeta_1(s) + 2^{-s}(1 + 3 \cdot 2^{-s})\zeta(s)\zeta(s)\zeta_1(s)$$

$$\zeta_{Pc}^{\triangleleft}(s) = (1 + 3 \cdot 2^{-s} - 4 \cdot 2^{-2s})\zeta(s)\zeta_1(s) + (3 \cdot 2^{-2s} + 2^{-s})\zeta(s)\zeta(s)\zeta_1(s)$$

$$\zeta_{Cm}^{\triangleleft}(s) = (3 \cdot 2^{-s} + 1)\zeta(s)\zeta_1(s) + 2^{-s} \cdot (1 - 2^{-s} + 4 \cdot 2^{-2s})\zeta(s)\zeta(s)\zeta_1(s)$$

$$\zeta_{Cc}^{\triangleleft}(s) = (1 - 2^{-s})\zeta(s)\zeta_1(s) + (4 \cdot 2^{-3s} - 2^{-2s} + 2^{-s})\zeta(s)\zeta(s)\zeta_1(s)$$

3 Applications: subgroups of a given index

In this and the following section, $d(n)$ denotes the number of all positive divisors of a positive integer n and $\sigma(n)$ denotes the sum of all positive divisors for a positive integer n , i. e. $\sigma(n) = \sum_{l|n} l$.

Proposition 3.1 The number of all subgroups of index n in the group $P\bar{1}$ is given by the following expressions

1. if n is even,

$$a_n = n \sum_{l|n} l \cdot \sigma(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l)$$

2. if n is odd, $a_n = n \sum_{l|n} l \cdot \sigma(l)$,

In particular, $a_p = p^3 + p^2 + p$ for every odd prime p .

Proposition 3.2 The number of all subgroups of index n in group $P2$ reads:

1. if n is even,

$$a_n = \sum_{l|n} l \cdot \sigma(l) + 8 \cdot \sum_{l|(\frac{n}{2})} l \cdot \sigma(l)$$

2. if n is odd, $a_n = \sum_{l|n} l \cdot \sigma(l)$,

In particular, if p is an odd prime, then $a_p = p^2 + p + 1$.

Proposition 3.3 *The number of all subgroups of index n in group $P2_1$ is $a_n = \sum_{l|n} l \cdot \sigma(l)$. In particular, $a_p = p^2 + p + 1$ for every odd prime p .*

Proposition 3.4 *The number of all subgroups of index n in the group $C2$ is*

$$a_n = \begin{cases} \sum_{l|n} l \cdot \sigma(l), & (n \equiv 1 \vee n \equiv 2 \vee n \equiv 3) \pmod{4} \\ \sum_{l|n} l \cdot \sigma(l) + 8 \cdot \sum_{l|(\frac{n}{4})} l \cdot \sigma(l), & n \equiv 0 \pmod{4} \end{cases}$$

In particular, if p is an odd prime, then $a_p = p^2 + p + 1$.

Proposition 3.5 *The number of all subgroups of index n in the group Pm is as follows:*

1. *if n is even,*

$$a_n = \begin{cases} \sum_{l|n} l \cdot d(l) + 9 \cdot \sum_{l|(\frac{n}{2})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 2 \pmod{4} \\ \sum_{l|n} l \cdot d(l) + 9 \sum_{l|(\frac{n}{2})} l \cdot d(l) + 6 \sum_{l|(\frac{n}{4})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 0 \pmod{4} \end{cases}$$

2. *if n is odd, $a_n = \sum_{l|n} l \cdot d(l)$.*

In particular, if p is an odd prime, then $a_p = 2p + 1$.

Proposition 3.6 *The number of all subgroups of index n in the group Pc is given by:*

1. *if n is even,*

$$a_n = \begin{cases} \sum_{l|n} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 2 \pmod{4} \\ \sum_{l|n} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot d(l) - 2 \cdot \sum_{l|(\frac{n}{4})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 0 \pmod{4} \end{cases}$$

2. *if n is odd, $a_n = \sum_{l|n} l \cdot d(l)$.*

In particular, if p is an odd prime, then $a_p = 2p + 1$.

Proposition 3.7 *The number of all subgroups of index n in the group Cm is:*

1. *if n is even,*

$$a_n = \begin{cases} \sum_{l|n} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8} \\ \sum_{l|n} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot d(l) + 6 \cdot \sum_{l|(\frac{n}{4})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 4 \pmod{8} \\ \sum_{l|n} ld(l) + \sum_{l|(\frac{n}{2})} ld(l) + 6 \sum_{l|(\frac{n}{4})} ld(l) + 8 \sum_{l|(\frac{n}{8})} ld(l) + \sum_{l|(\frac{n}{2})} l\sigma(l), & n \equiv 0 \pmod{8} \end{cases}$$

2. if n is odd, $a_n = \sum_{l|n} l \cdot d(l)$.

In particular, if p is an odd prime, then $a_p = 2p + 1$.

Proposition 3.8 *The number of all subgroups of index n in the group Cc is the following:*

1. if n is even,

$$a_n = \begin{cases} \sum_{l|n} l \cdot d(l) - 3 \cdot \sum_{l|(\frac{n}{2})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8} \\ \sum_{l|n} l \cdot d(l) - 3 \cdot \sum_{l|(\frac{n}{2})} l \cdot d(l) + 10 \cdot \sum_{l|(\frac{n}{4})} l \cdot d(l) + \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 4 \pmod{8} \\ \sum_{l|n} ld(l) - 3 \sum_{l|(\frac{n}{2})} ld(l) + 10 \sum_{l|(\frac{n}{4})} ld(l) - 8 \sum_{l|(\frac{n}{8})} ld(l) + \sum_{l|(\frac{n}{2})} l\sigma(l), & n \equiv 0 \pmod{8} \end{cases}$$

2. if n is odd, $a_n = \sum_{l|n} l \cdot d(l)$.

In particular, if p is an odd prime, then $a_p = 2p + 1$.

4 Applications: normal subgroups of a given index

Proposition 4.1 *The number of all normal subgroups of index n in the group $P\bar{1}$ reads:*

1. $c_1 = 1$,

2. if n is odd and $n \neq 1$, then $c_n = 0$,

$$3. \text{ if } n \text{ is even, } c_n = \begin{cases} 15, & n = 2, \\ 35, & n = 4, \\ 43, & n = 8, \\ \sum_{l|(\frac{n}{2})} l \cdot \sigma(l), & n \equiv 0 \pmod{2} \wedge n \neq 1, 2, 4, 8 \end{cases}$$

Proposition 4.2 *The number of all normal subgroups of index n in the group $P2$ is:*

1. if n is even,

$$c_n = \begin{cases} 40 + 3 \cdot \sum_{l|(\frac{n}{4})} \sigma(l) + \sum_{l|(\frac{n}{2})} \sigma(l), & n \equiv 0 \pmod{8}, \\ 14 + \sum_{l|(\frac{n}{2})} \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8} \\ 36 + 3 \cdot \sum_{l|(\frac{n}{4})} \sigma(l) + \sum_{l|(\frac{n}{2})} \sigma(l), & n \equiv 4 \pmod{8} \end{cases}$$

2. if n is odd, $c_n = 1$.

Proposition 4.3 *The number of all normal subgroups of index n in the group $P2_1$ is given by:*

1. if n is even,

$$c_n = \begin{cases} 3 \cdot \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 0 \pmod{8}, \\ 6 + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8}, \\ 4 + 3 \cdot \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 4 \pmod{8} \end{cases}$$

2. if n is odd, $c_n = 1$.

Proposition 4.4 *The number of all normal subgroups of index n in the group $C2$ is:*

1. if n is even,

$$c_n = \begin{cases} 6 + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8} \\ 8 - \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 4 \pmod{8} \\ 8 + 4 \cdot \sum_{l|\left(\frac{n}{8}\right)} \sigma(l) - \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 0 \pmod{8} \end{cases}$$

2. if n is odd, $c_n = 1$.

Proposition 4.5 *The number of all normal subgroups of index n in the group Pm is:*

1. if n is even,

$$c_n = \begin{cases} \sigma(n) + 11 \cdot \sigma\left(\frac{n}{2}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 2 \pmod{4} \\ \sigma(n) + 11 \cdot \sigma\left(\frac{n}{2}\right) + 12 \cdot \sigma\left(\frac{n}{4}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l) + 3 \cdot \sum_{l|\left(\frac{n}{4}\right)} \sigma(l), & n \equiv 0 \pmod{4} \end{cases}$$

2. if n is odd, $c_n = \sigma(n)$.

In particular, if p is an odd prime, then $c_p = p + 1$.

Proposition 4.6 *The number of all normal subgroups of index n in the group Pc reads:*

1. if n is even,

$$c_n = \begin{cases} \sigma(n) + 3 \cdot \sigma\left(\frac{n}{2}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 2 \pmod{4} \\ \sigma(n) + 3 \cdot \sigma\left(\frac{n}{2}\right) - 4 \cdot \sigma\left(\frac{n}{4}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l) + 3 \cdot \sum_{l|\left(\frac{n}{4}\right)} \sigma(l), & n \equiv 0 \pmod{4} \end{cases}$$

2. if n is odd, $c_n = \sigma(n)$.

In particular, if p is an odd prime, then $c_p = p + 1$.

Proposition 4.7 *The number of all normal subgroups of index n in the group Cm is given by:*

1. if n is even,

$$c_n = \begin{cases} \sigma(n) + 3 \cdot \sigma\left(\frac{n}{2}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8} \\ \sigma(n) + 3 \cdot \sigma\left(\frac{n}{2}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l) - \sum_{l|\left(\frac{n}{4}\right)} \sigma(l), & n \equiv 4 \pmod{8} \\ \sigma(n) + 3 \cdot \sigma\left(\frac{n}{2}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l) - \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + 4 \cdot \sum_{l|\left(\frac{n}{8}\right)} \sigma(l), & n \equiv 0 \pmod{8} \end{cases}$$

2. if n is odd, $c_n = \sigma(n)$.

In particular, if p is an odd prime, then $c_p = p + 1$.

Proposition 4.8 *The number of all normal subgroups of index n in the group Cc is the following:*

1. if n is even,

$$c_n = \begin{cases} \sigma(n) - \sigma\left(\frac{n}{2}\right) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & (n \equiv 2 \vee n \equiv 6) \pmod{8} \\ \sigma(n) - \sigma\left(\frac{n}{2}\right) - \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l), & n \equiv 4 \pmod{8} \\ \sigma(n) - \sigma\left(\frac{n}{2}\right) - \sum_{l|\left(\frac{n}{4}\right)} \sigma(l) + \sum_{l|\left(\frac{n}{2}\right)} \sigma(l) + 4 \cdot \sum_{l|\left(\frac{n}{8}\right)} \sigma(l), & n \equiv 0 \pmod{8} \end{cases}$$

2. if n is odd, $c_n = \sigma(n)$.

In particular, if p is an odd prime, then $c_p = p + 1$.

5 Proof

The method for calculating the number of subgroups of any given index in a group G having an abelian normal subgroup T of a finite index (see [6]) could be applied to any polycyclic group or to any poly-(infinite) cyclic-by-finite group. In the sequel, G will denote a space group with the point group isomorphic to the cyclic group of order 2. In each particular case, we make two standard steps. We firstly count all subgroups containing T as its subgroup. Then we count all subgroups contained in T .

In our setting, G is a finite extension of a free abelian group T of rank 3. Group T is generated by three translations x , y and z . A subgroup of finite index in T is free and generated by elements $x^a y^b z^c$, $y^d z^e$ and z^f . These exponents are unique to the limits: $a, d, f > 0$, $0 \leq b < d$, $0 \leq c, e < f$. Since P is a group isomorphic to the cyclic group of order 2, the index of a subgroup of T in group G is $2adf$. We know that the zeta function of $T \cong \mathbb{Z}^3$ is $\zeta(s)\zeta(s-1)\zeta(s-2)$. Therefore, the contribution to the zeta function of a group G coming from this part of the problem is $2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ for all eight groups.

Thus, we only need to count subgroups containing T . So, let H_1 be a subgroup containing T as its subgroup. Then H_1 is generated by elements $rx^a y^b z^c$, $x^d y^e z^f$, $y^g z^h$ and z^i , where x , y and z are generators of T and rT is a generator of the point group P , which is cyclic of order 2. Furthermore, these exponents are unique to the limits: $d, g, i > 0$, $0 \leq a < d$, $0 \leq b, e < g$, $0 \leq c, f, h < i$. The index of a subgroup generated by these elements is dgi . Since T is a normal subgroup in G , then $H_1 \cap T$ is a normal subgroup in T and

$$H_1 / (H_1 \cap T) \simeq H_1 T / T.$$

This means that $(x^d y^e z^f)^r$, $(y^g z^h)^r$ and $(z^i)^r$ are elements of $H_1 \cap T$. In this case

$$H_1 T / T \simeq P,$$

hence $(rx^a y^b z^c)^2$ is an element of $H_1 \cap T$. The problem of counting subgroups is reduced to solving the system of equations derived from these conditions. In effect, we consider the number of possible combinations of values (solutions of the corresponding system of equations) which the exponents of the generators of H_1 may take.

If H_1 is a normal group in G , then its elements have also to satisfy relations $(rx^a y^b z^c)^r$, $(rx^a y^b z^c)^x$, $(rx^a y^b z^c)^y$, $(rx^a y^b z^c)^z \in H_1$. Since normality is not a transitive relation, we must also add conditions which will ensure that a normal subgroup of T is a normal subgroup of G . Let $H_2 = \langle x^a y^b z^c, y^d z^e, z^f \rangle$ be a normal group of T . Then H_2 is a normal subgroup of G , if $(x^a y^b z^c)^r$, $(y^d z^e)^r$ and $(z^f)^r$ are elements of H_2 .

When writing a space group in an abstract form, we follow the descriptions of these groups given in [5]. The software packages Mathematica Wolfram and GAP were apt for double checking the calculations. Mathematica was used to convert our formulas into lists of integers a_n or c_n .

We demonstrate the above technique in detail in the case of group $C2$.

5.1 Zeta function of group C2

Recall that $C2 = \langle x, y, z, r \mid [x, y], [x, z], [y, z], r^2, x^r = xy, y^r = y^{-1}, z^r = z^{-1} \rangle$.

We are counting subgroups of the form $H_1 = \langle rx^a y^b z^c, x^d y^e z^f, y^g z^h, z^i \rangle$. Each of $(rx^a y^b z^c)^2, (x^d y^e z^f)^r, (y^g z^h)^r, (z^i)^r$ must lie in $H_1 \cap T$. Now, $(rx^a y^b z^c)^2 = x^{2a} y^a, (x^d y^e z^f)^r = x^{-d} y^{-e} z^{-f} \cdot x^{2d} y^d, (y^g z^h)^r = y^{-g} z^{-h}, (z^i)^r = z^{-i}$. Each of $x^{-d} y^{-e} z^{-f}, y^{-g} z^{-h}, z^{-i}$ is contained in $H_1 \cap T = \langle x^d y^e z^f, y^g z^h, z^i \rangle$, regardless of the values of d, e, f, g, h, i . So, $(rx^a y^b z^c)^2$ lies in $H_1 \cap T$ if $x^{2a} y^a$ lies in $H_1 \cap T$; $(x^d y^e z^f)^r$ lies in $H_1 \cap T$ if $x^{2d} y^d$ lies in $H_1 \cap T$. If $x^{2a} y^a, x^{2d} y^d$ are in $H_1 \cap T$ then there exist integers $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ such that:

$$x^{2d} y^d = (x^d y^e z^f)^{\alpha_1} (y^g z^h)^{\beta_1} (z^i)^{\gamma_1}, x^{2a} y^a = (x^d y^e z^f)^{\alpha_2} (y^g z^h)^{\beta_2} (z^i)^{\gamma_2}.$$

We get the following system of equations:

$$S_1 = \left\{ \begin{array}{l} d\alpha_1 = 2d, e\alpha_1 + g\beta_1 = d, f\alpha_1 + h\beta_1 + i\gamma_1 = 0, \\ d\alpha_2 = 2a, e\alpha_2 + g\beta_2 = a, f\alpha_2 + h\beta_2 + i\gamma_2 = 0, \end{array} \right\}.$$

The first equation implies $\alpha_1 = 2$. Consider the equation $2e = -g\beta_1 + d$. The left side of the equation is even. If g is even, then d has also to be even. We get: $\frac{d}{g} \geq \beta_1 > -2 + \frac{d}{g}$. There are two integers in the interval $\left[\frac{d}{g}, \frac{d}{g} - 2\right)$. To solve the above system of equations, we shall consider the following cases:

Case 1. d, g, i are odd. In this case, there is one choice for a . Since β_1 has to be odd, there is one choice for e . From $2f + h\beta_1 + i\gamma_1 = 0$, we get $-\frac{h\beta_1}{i} \geq \gamma_1 > -2 - \frac{h\beta_1}{i}$. Now, $h\beta_1$ and $i\gamma_1$ have to be odd or even at the same time. Since β_1 and i are odd, we conclude that if h is odd then γ_1 is odd and if h is even then γ_1 is even. Hence, there are i choices for h and one choice for f . The zeta function contribution in this case is: $\sum_{d,g,i \in \mathbb{N}} d^{-s} g^{-s} i^{-s} gii$, where $\mathbb{N}' = \{2k - 1 \mid k \in \mathbb{N}\}$.

Case 2. and Case 3. (d, i are odd, g is even) and (d is odd, g, i are even). These cases are impossible.

Case 4. i, g are odd, d is even. There are two choices for a . Since β_1 has to be even, there is one choice for e . Furthermore, γ_1 has to be even, so there is one choice for f . The zeta function contribution in this case is: $2 \cdot \sum_{d \in 2\mathbb{N}, g, i \in \mathbb{N}'} d^{-s} g^{-s} i^{-s} gii$, where $\mathbb{N}' = \{2k - 1 \mid k \in \mathbb{N}\}$.

Case 5. g, d are odd, i is even. There is one choice for a ; β_1 has to be odd, so there is one choice for e . Since $h\beta_1$ and $i\gamma_1$ have to be odd or even at the same time, we see that h has to be even. There are two choices for γ_1 , hence there are two choices for f . The zeta function contribution in this case is: $\sum_{i \in 2\mathbb{N}, g, d \in \mathbb{N}'} d^{-s} g^{-s} i^{-s} gi \cdot \frac{i}{2} \cdot 2$.

Case 6. g is odd, i, d are even. There are two choices for a ; β_1 has to be even, so there is one choice for e . If $a = 0$, then there are two choices for f ; if $a = \frac{d}{2}$ then γ_1 is

even, so there is one choice for f in this case. The zeta function contribution reads: $3 \sum_{g \in \mathbb{N}', d, i \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} gii$.

Case 7. g, d are even, i is odd. There are two choices for a . If $a = 0$ and β_1 is even, then γ_1 is even and there are i choices for h . Hence, for $a = 0$ and β_1 is even, there are one choice for e and one choice for f . If $a = 0$ and β_1 is odd, then γ_1 and h are both even or both odd. If $a = \frac{d}{2}$ then β_1, γ_1 are even and there is one choice for f . The zeta function contribution in this case is: $3 \sum_{i \in \mathbb{N}', d, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} gii$.

Case 8. g, d, i are even. There are two choices for a . If $a = 0$ and β_1 is even, then there are two choice for γ_1 and there are i choices for h . If $a = 0$ and β_1 is odd, then there are two choices for γ_1 and h is even ($\frac{i}{2}$ choices for h). If $a = \frac{d}{2}$ then β_1, γ_1 are even and there is one choice for f . The zeta function contribution in this case is:

$$\begin{aligned} & \sum_{d, i, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} \cdot g \cdot i \cdot i \cdot 2 + \sum_{d, i, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} \cdot g \cdot i \cdot \frac{i}{2} \cdot 2 \\ & + \sum_{d, i, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} \cdot g \cdot i \cdot i = 4 \sum_{d, i, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} \cdot g \cdot i \cdot i. \end{aligned}$$

Finally, we obtain the zeta function for group C_2 :

$$\begin{aligned} \zeta_{C_2}(s) &= \sum_{d, g, i \in \mathbb{N}'} d^{-s} g^{-s} i^{-s} gii + 2 \cdot \sum_{d \in 2\mathbb{N}, g, i \in \mathbb{N}'} d^{-s} g^{-s} i^{-s} gii + \\ &+ \sum_{d, g \in \mathbb{N}', i \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} gii + 3 \sum_{g \in \mathbb{N}', i, d \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} gii \\ &+ 3 \sum_{i \in \mathbb{N}', d, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} gii + 4 \sum_{d, i, g \in 2\mathbb{N}} d^{-s} g^{-s} i^{-s} gii \\ &+ 2^{-s} \zeta(s) \zeta(s-1) \zeta(s-2) \\ &= (1 + 2^{-2s+3}) \zeta(s) \zeta_1(s) \zeta_2(s). \end{aligned}$$

5.2 Normal zeta function of group C_2

We use the set of constraints which we obtained in the previous Subsection. By counting the number of subgroups $H_1 = \langle r x^a y^b z^c, x^d y^e z^f, y^g z^h, z^i \rangle$ of C_2 , we deduced the system S_1 .

Based on the conditions of normality, we get another set of constraints:

$$S_2 = \left\{ \begin{aligned} d\alpha_3 &= 0, e\alpha_3 + g\beta_3 = -1, f\alpha_3 + h\beta_3 + i\gamma_3 = 0 \\ d\alpha_4 &= 0, e\alpha_4 + g\beta_4 = 2, f\alpha_4 + h\beta_4 + i\gamma_4 = 0, \\ d\alpha_5 &= 0, e\alpha_5 + g\beta_5 = 0, f\alpha_5 + h\beta_5 + i\gamma_5 = 2 \end{aligned} \right\}.$$

The equations $d\alpha_3 = 0, e\alpha_3 + g\beta_3 = -1, f\alpha_3 + h\beta_3 + i\gamma_3 = 0$ imply that $g = 1, \beta_3 = -1$, so $b = e = 0 = h$, while the equations $d\alpha_5 = 0, e\alpha_5 + g\beta_5 =$

0, $f\alpha_5 + h\beta_5 + i\gamma_5 = 2$ imply that $i = 1$ or $i = 2$. Observing four cases depending on whether d is even or odd and depending on values of i , we get

$$\sum_{d \in \mathbb{N}'} d^{-s} + 2 \sum_{d \in 2\mathbb{N}} d^{-s} + 4 \cdot 2^{-s} \sum_{d \in \mathbb{N}} d^{-s} + 6 \sum_{d \in 2\mathbb{N}} d^{-s} 2^{-s} = (2 \cdot 2^{-2s} + 5 \cdot 2^{-s} + 1)\zeta(s).$$

Now, we count normal subgroups of T . Any such subgroup takes the form $H_2 = \langle x^a y^b z^c, y^d z^e, z^f \rangle$. We assume $0 < a, 0 \leq b < d, 0 \leq c, e < f$. Let us check the conditions of normality in C_2 . These require that $(x^a y^b z^c)^r, (y^d z^e)^r, (z^f)^r$ are in H_2 . After some calculations, we get the next set of constraints:

$$S_3 = \{a\alpha_1 = 2a, b\alpha_1 + d\beta_1 = a, c\alpha_1 + e\beta_1 + f\gamma_1 = 0\}.$$

This system of equations reduces to three equations: $d\beta_1 = a - 2b, e\beta_1 + f\gamma_1 = 0, -2e + f\gamma_1 = 0$. The case a odd and d even is impossible. Thus, we have six cases depending on whether a or d or f are odd or even. The respective contributions are:

$$\begin{aligned} & \sum_{a,d,f \in \mathbb{N}'} a^{-s} d^{-s} f^{-s} f + \sum_{a \in 2\mathbb{N}, d, f \in \mathbb{N}'} a^{-s} d^{-s} f^{-s} f \\ & + 2 \sum_{a, f \in 2\mathbb{N}, d \in \mathbb{N}'} a^{-s} d^{-s} f^{-s} f + 2 \sum_{a, d \in 2\mathbb{N}, f \in \mathbb{N}'} a^{-s} d^{-s} f^{-s} f \\ & + \sum_{a, d \in \mathbb{N}', f \in 2\mathbb{N}} a^{-s} d^{-s} f^{-s} f + 3 \sum_{a, d, f \in 2\mathbb{N}} a^{-s} d^{-s} f^{-s} f \\ & = (1 - 2^{-s} + 4 \cdot 2^{-2s})\zeta(s)\zeta(s)\zeta(s-1). \end{aligned}$$

Combining the contributions coming from H_1 and H_2 , we get the normal zeta function of C_2 :

$$\zeta_{C_2}^{\triangleleft}(s) = (2 \cdot 2^{-2s} + 5 \cdot 2^{-s} + 1)\zeta(s) + 2^{-s} \cdot (1 - 2^{-s} + 4 \cdot 2^{-2s})\zeta(s)\zeta(s)\zeta_1(s).$$

6 Remarks

As far as Sect. 5.1 is concerned, using a similar argumentation one can obtain more general assertions of the following form.

Proposition 6.1 *Let $n \geq 2$ be a fixed integer and G_n be the group defined by*

$$G_n = \left\langle r, x_1, x_2, \dots, x_n \mid r^2, [x_j, x_k] (\forall 1 \leq j, k \leq n), x_i^r = x_i^{-1} (\forall 1 \leq i \leq n) \right\rangle.$$

The zeta function of group G_n is given by

$$\zeta_{G_n}(s) = \zeta(s-1)\zeta(s-2) \cdots \zeta(s-n) + 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2) \cdots \zeta(s-n+1).$$

Proposition 6.2 *Let $n \geq 2$ be an integer and G_n be the group defined by $G_n = \langle r, x_1, x_2, \dots, x_n \mid r^2, [x_j, x_k] (\forall 1 \leq j, k \leq n), x_1^r = x_1 x_2, x_i^r = x_i^{-1} (\forall 1 < i \leq n) \rangle$. The zeta function of group G_n is $\zeta_{G_n}(s) = (1 + 2^{-2s+n})\zeta(s)\zeta(s-1)\zeta(s-2) \cdots \zeta(s-n+1)$. The zeta function of group G_n has an Euler product.*

References

1. M.P.F. du Sautoy, A. Lubotzky, Functional equations and uniformity for local zeta functions of nilpotent groups. *Am. J. Math.* **118**(4), 39–90 (1996)
2. M.P.F. du Sautoy, J.J. McDermott, G.C. Smith, Zeta functions of crystallographic groups and analytic continuation. *Proc. Lond. Math. Soc.* **79**, 511–534 (2010)
3. M.P.F. du Sautoy, L. Woodward, in *Zeta Functions of Groups and Rings*. Lecture Notes in Mathematics 1925, (Springer, 2008)
4. F.J. Grunewald, D. Segal, G.C. Smith, Subgroups of finite index in nilpotent groups. *Invent. Math.* **93**, 185–223 (1988)
5. E. Lord, *Generators and relations for space groups*. Bangalore (2010)
6. J.J. McDermott, *Subgroup growth and the plane crystallographic groups*. Ph.D., University of Bath, UK (1997)
7. G.C. Smith, *Zeta-functions of torsion-free finitely generated nilpotent groups*. Ph.D., University of Manchester, UK (1983)
8. C. Voll, Functional equations for zeta functions of groups and rings. *Ann. Math.* **172**, 1181–1218 (2010)
9. C. Voll, A. Beauville, Functional equations for local normal zeta functions of nilpotent groups. *Geom. Funct. Anal. GAFA* **15**, 274–295 (2005)